Shock wave dynamics in a discrete nonlinear Schrödinger equation with internal losses

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Propagation of a shock wave (SW), converting an energy-carrying domain into an empty one, is studied in a discrete version of the normal-dispersion nonlinear Schrödinger equation with viscosity, which may describe, e.g., an array of optical fibers in a weakly lossy medium. It is found that the SW in the discrete model is stable, as well as in its earlier studied continuum counterpart. In a strongly discrete case, the dependence of the SWs velocity upon the amplitude of the energy-carrying background is found to obey a simple linear law, which differs by a value of the proportionality coefficient from a similar law in the continuum model. For the underdamped case, the velocity of the shock wave is found to be vanishing along with the viscosity constant. We argue that the latter feature is universal for long but finite systems, both discrete and continuum. The dependence of the SW's width on the parameters of the system is also discussed.

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I. INTRODUCTION

The discrete nonlinear Schrödinger equation (NLS), also called the discrete self-trapping equation [1], is a well-known dynamical model with numerous physical applications ranging from nonlinear optics to the theory of molecular vibrations [2]. Unlike its celebrated continuum counterpart, the NLS equation proper, the discrete NLS equation is not integrable for systems with more than two sites (as, besides the Hamiltonian, the model has only one more conserved quantity), and it does not have exact soliton solutions. In spite of this, it demonstrates a rich variety of dynamical behaviors, including bright and dark solitary waves and shock waves (SWs), i.e., sharp expanding fronts followed by localized excitations and background oscillations [3]. These dynamical states are typical for Hamiltonian discrete NLS-like systems and were found in several physical contexts such as chains of two-level atoms with excitons, one-dimensional (1D) ferromagnetic Heisenberg magnets, Toda chains, etc. [4–6]. In non-Hamiltonian (dissipative) systems, however, a different type of SW can exist, viz., a steadily moving front separating two different states (phases), a trivial (zero-amplitude) one, and a finite-amplitude continuous wave (cw). To distinguish this type of SW, we will refer to them as SWs of a kink type. In contrast with the SWs observed in Hamiltonian lattices, these SWs exist also in continuum models, such as the Burgers equation, NLS equations with losses, which are of interest both for applications and by themselves [7–10]. In particular, there has recently been interest in physically relevant modifications of the NLS equation that describes the light propagation in optical fibers [8,9] which may give rise to stable SWs of the kink type. The simplest modified NLS equation that supports them is the one considered in Ref. [9]: it is the usual NLS equation with the *normal dispersion* (which is necessary to make the cw state modulationally stable), to which a single additional term, describing intrinsic dispersive (diffusionlike) losses, is added,

$$iu_z+(1/2)u_{tt}-|u|^2u=i\alpha u_{tt}. \tag{1}$$

This equation is written in the standard ''fiber'' notation [11], with z, t, and u standing for, respectively, the propagation distance, *reduced time*, and envelope of the electromagnetic field in the optical fiber. As concerns the physical origin of the term on the right-hand side of Eq. (1), dissipative losses in optical fiber are represented by the term $-i\gamma u$ on the same side of the equation ($\gamma>0$ is an attenuation constant). The losses can be compensated by optical amplifiers, providing for a *bandwidth-limited* gain which may be approximated by terms $i\beta u+i\alpha u_{tt}$, with positive β and α [11]. To exactly compensate the attenuation, one chooses $\beta=\gamma$. The dispersive loss factor α that remains in Eq. (1) accounts both for the naturally limited gain bandwidth of the amplifier, and for the possible additional reduction of the bandwidth induced by optical filters which are used to suppress noise in optical communication lines [11]. As it was demonstrated analytically and numerically in Ref. [9], Eq. (1) with $\alpha>0$ always gives rise to a stable SW of the form $u(z,t)=U(t-Vz)\exp(-i\rho^2z)$, where the complex function U takes boundary values $U(-\infty)=0, U(+\infty)=\rho,\rho$ being the asymptotic cw amplitude, and V is the (inverse) SWs velocity. The SW exists only with $V>0$ corresponding to a situation when the zero-state domain expands into the energy-carrying one.

The aim of the present paper is to show that stable SWs of the kink type exist as well in a dissipative version of the discrete NLS equation, and to investigate their basic properties, such as the dependence of the velocity and the width of the wave upon various parameters. We note that similar so-

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lutions have also been found in a modified Toda lattice with intrinsic losses which is, as a matter of fact, a discrete version of the well-known Korteweg–de Vries Burgers equation [10]. In models of that type, however, the SWs velocity does *not* depend on the dissipative constant, being determined solely by the asymptotic values of the field at $\pm\infty$ [10]. On the contrary, we will demonstrate that SWs in the discrete dissipative NLS equation move at a constant velocity which explicitly depends on the damping parameter. An expression for the velocity is also obtained by means of the power balance applied to the second conserved quantity of the system, i.e., the ''norm'' (also called energy in nonlinear optics). This expression, except for the very small damping region, is found to be in excellent agreement with numerical results.

We remark that the physical realization of our model can be given in terms of an array of nonlinear optical fibers, n being the fiber's number. The dissipative coupling implies that the fiber array is placed into a weakly lossy medium. A particular example is a configuration in which the array is parallel to a planar waveguide, in which light can create free charge carriers that are subject to diffusion, which takes place in a semiconductor waveguide.

The paper is organized as follows. In Sec. II we formulate the discrete model and discuss some of its properties. In Sec. III we display typical examples of the SWs produced by simulations. Section IV is devoted to the analysis of the dependence of the SWs velocity on the asymptotic cw amplitude ρ and on the viscosity α, which are basic characteristic of the SW. We compare in detail these dependences in the discrete and continuum versions of the model. Section V concludes the paper, and a proof of an important statement, that the velocity of the SW-like profile in the discrete NLS equation may only be zero at $\alpha=0$, is given in the Appendix.

II. FORMULATION OF THE PROBLEM

It is a subject of principal interest to understand if SWs exist in a discrete version of the NLS equation including intrinsic losses, which can prevent the development of dynamical chaos (observed in the absence of damping) and stabilize a SW. The simplest version of such a model is obtained by the direct discretization of Eq. (1),

$$i\dot{u}_n+(1-i\alpha)(u_{n+1}+u_{n-1}-2u_n)-|u_n|^2u_n=0, \tag{2}$$

where the evolution variable is now t, the overdot standing for d/dt, and the coefficient in front of the dispersive term is 1 instead of 1/2. This equation is supplemented by the boundary conditions (b.c.) coinciding with those in the continuum model described above,

$$\lim_{n\to-\infty} u_n=0, \qquad \lim_{n\to+\infty} u_n=\rho\exp(-i\rho^2 t). \tag{3}$$

For an array of nonlinear optical fiber in a weakly lossy medium, t has the meaning of propagation distance along each fiber and the conservative linear coupling accounts for tunneling of light between adjacent parallel fibers [12].

To demonstrate that the terms $\sim\alpha$ in Eq. (2) [as well as in Eq. (1)], are strictly dissipative, one can define the norm (''number of quanta'')

$$E=\sum_{n=-\infty}^{+\infty}[|u_n|^2-|u_n(t=0)|^2], \tag{4}$$

and the Hamiltonian

$$H=\sum_{n=-\infty}^{+\infty}\Big\{(u_{n+1}\overline{u_n}+\overline{u_{n+1}}u_n-2|u_n|^2) -\frac{1}{2}[|u_n|^4-|u_n(t=0)|^4]\Big\} \tag{5}$$

of the conservative version of the model. Here, the variables u_n without an explicitly written argument pertain to a given moment of time, while $\overline{u_n}$ is its complex conjugate. The terms pertaining to $t=0$ are subtracted in Eqs. (4) and (5) to suppress a trivial divergence at $|n|\to\infty$. Both E and H are conserved if $\alpha=0$. When $\alpha>0$, a straightforward consideration leads to the following exact evolution equation for E:

$$\frac{dE}{dt}=-2\alpha\sum_{n=-\infty}^{+\infty}|u_n-u_{n+1}|^2. \tag{6}$$

Thus, for all the configurations, except for the cw state with all u_n identical, the energy may only decrease in time. In contrast to the continuum case and integrable discrete models like the Toda lattice, our discrete model does not have an exact steadily moving solution (see also Ref. [14]). On the other hand, numerical simulations show (see below) that SWs of the kink type can be easily formed from initial conditions in the form of a step function separating two regions: an energy-carrying region ($u_n=\rho$ for $n>n_0$) and a zero-energy region ($u_n=0$ for $n<n_0$). The propagation of the shock will destroy the background just like the flame devours the wax during its motion along a candle. In analogy with the Faraday's candle theory [13,2], the SW will move at a constant velocity determined by a power-balance condition. Indeed, by adopting the norm (4) as the energy, we conclude that the velocity is a ratio between the power dissipated as per Eq. (6) and the energy density ρ^2 stored in the background, i.e.,

$$V=\rho^{-2}\frac{dE}{dt}=-2\alpha\rho^{-2}\sum_{n=-\infty}^{+\infty}|u_n-u_{n+1}|^2. \tag{7}$$

From this expression it is clear that the power balance velocity of the SW vanishes in the case $\alpha=0$. This complies with simulations displayed in Fig. 1: in the absence of the viscosity, the front spreads out into a chaotic pattern with no motion of the profile as a whole. In the Appendix we shall provide for another argument towards the nonexistence of moving SWs in the absence of dissipation. The spreading out of the front in the case $\alpha=0$ can be easily explained by the action of the conservative finite-difference operator in Eq. (2). Indeed, on the finite background with the intensity ρ, the dispersion law for small perturbations with a wave number k gives rise to the group velocity

$$v_{\rm gr}=\frac{\rho^2+2\sin^2(k/2)}{\sqrt{\rho^2+\sin^2(k/2)}}\cos(k/2). \tag{8}$$

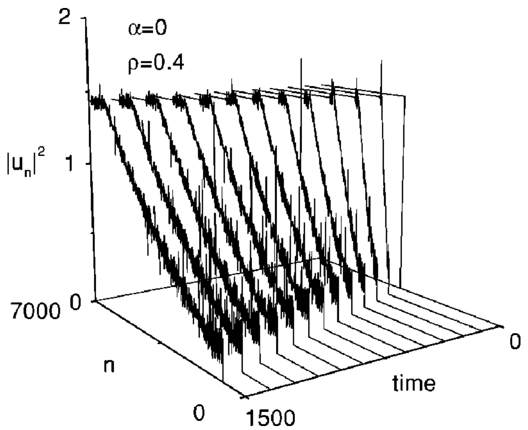

FIG. 1. Numerically simulated decay of the initial step in Eq. (2) with $\alpha=0$ and $\rho=0.4$ (plotted quantities are dimensionless).

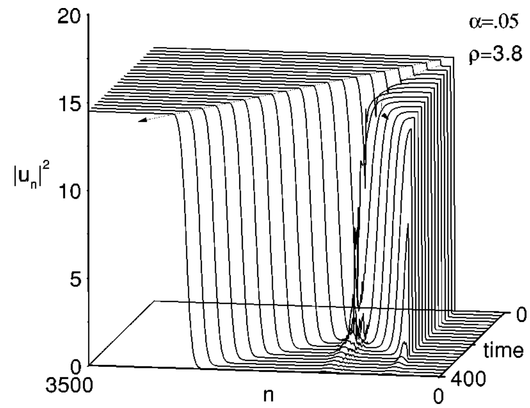

FIG. 3. The same as in Fig. 2 with $\alpha=0.05$ and $\rho=3.8$. In this case the estimate (10) predicts a relatively small width of the shock wave, $W\sim 5$, in accord with which the formation of a steep, essentially discrete, shock wave is observed. Plotted quantities are dimensionless.

Due to the difference between these values on the two sides of the SW configuration (finite ρ and $\rho=0$), one could expect that the spreading-out of the front at the upper level occurs more rapidly than at the zero level. This is what we indeed observe in the simulations of the case $\alpha=0$ displayed in Fig. 1, and also at a transient stage of the evolution in the weakly dissipative case in Fig. 2.

III. NUMERICAL SIMULATIONS OF THE SHOCK WAVE

To identify the cases in which the model (2) is essentially discrete, it is useful to refer to estimates for the SWs width and velocity derived in Ref. [9]. According to the estimates, for small and large α the velocity of the SW of the continuum model can be approximated by

$$V_{\text{contin}}\approx\begin{cases}\sqrt{2}\rho & \text{at} \quad \alpha\rho\ll 1\\ C_{\text{contin}}\,\rho\sqrt{\alpha} & \text{at} \quad \alpha\rho\gg 1,\end{cases} \tag{9}$$

where the factor $\sqrt{2}$ takes into regard the difference in the definition of the dispersion coefficient in Eqs. (1) and (2). The constant C_{contin}, corresponding to the continuum *overdamped* model, is ~ 1 (its exact value was not reported in Ref. [9]; in this work, it will be found that, in the present notation, C_{contin} is very close to 0.75). In Ref. [9] it was also shown that the size of the shock scales as

$$W_{\text{contin}}\sim\begin{cases}(\alpha\rho)^{-1} & \text{at} \quad \alpha\rho\ll 1\\ \sqrt{\alpha}\rho^{-1} & \text{at} \quad \alpha\rho\gg 1.\end{cases} \tag{10}$$

If the width of the shock is large enough, one may expect that the discrete model is well approximated by the continuum one, but the dynamics in the discrete model may be quite different if Eq. (10) predicts a (relatively) small width (say, $\lesssim 5$). As it is seen from Eq. (10), taking the limits of both small and large α drives the system out of the region where it is essentially discrete, which is well corroborated by our simulations of Eq. (2). In the low-amplitude limit, $\rho\to 0$, the SW becomes very broad also according to Eq. (10). However, in the opposite limit, $\rho\to\infty$, the SW becomes narrow, thus the study of the discrete model should focus on this case. In particular, if both α and ρ are large, the essentially discrete case corresponds to $\rho\gg\sqrt{\alpha}$. It is relevant to mention that, in the quasicontinuum limit, one can approximate the finite-difference operator in Eq. (2) by $u_{n+1}+u_{n-1}-2u_n \approx\partial^2 u/\partial n^2+(1/12)\partial^4 u/\partial n^4$, treating n as a continuous coordinate and taking into regard the fourth-order correction. Substituting this approximation into Eq. (2), it is possible to calculate the SWs velocity in the resulting fourth-order equation for the underdamped limit $\alpha\rho\ll 1$. Without displaying technical details, the result is the same as for the second-order equation (1), i.e., $V_{\text{contin}}=\sqrt{2}\rho$, see Eq. (9). We also notice that, in the underdamped case, smooth excitations of a different type, involving many lattice sites, may exist at large values of ρ [3]. Lastly, when the group-velocity dispersion for long waves corresponding to the limit of small k in Eq. (8) vanishes (i.e., at $\rho=\sqrt{3}$), SWs different from those considered in the present work are possible, too [4].

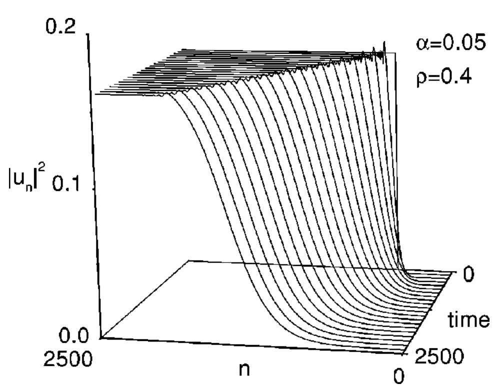

FIG. 2. Formation of a shock wave from the initial step in Eq. (2) with $\alpha=0.05$ and $\rho=0.4$. In this case the estimate (10) predicts a large width of the shock wave, $W\sim 50$, i.e., this is a quasicontinuum case. Plotted quantities are dimensionless.

In Figs. 1–4, we display typical examples of the SW found in direct simulations of Eq. (2). The equation was solved numerically using a fourth-order Runge-Kutta scheme in the region $0\leqslant n\leqslant N$ with periodic boundary conditions, the lattice size N taking values up to 5000 to avoid the influence of the system's finite size. The accuracy of the integration scheme was checked in the case $\alpha=0$ by controlling the conservation of both the Hamiltonian and the norm. The initial configuration was taken as a step between two domains with $u=0$ and $u=\rho$. As was mentioned above, in

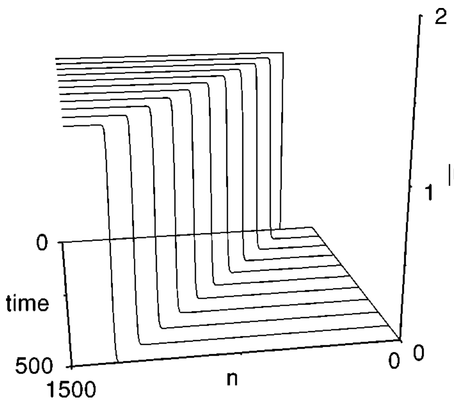

FIG. 4. The same as in Fig. 2 with $\alpha=2.0$ and $\rho=1.2$. In this case, the estimate (10) yields the width $W\sim 1$, and a very steep shock wave is indeed established almost instantaneously. Plotted quantities are dimensionless.

the absence of the intrinsic losses, $\alpha=0$ (Fig. 1), the initial step quickly spreads out, generating local chaotic oscillations. With $\alpha>0$, a broad SW close to that existing in the continuum model is indeed established, after a transient process, in the case when the estimates (10) predict $W_{\mathrm{contin}}\gg 1$ (Fig. 2). Contrary to that, an essentially discrete case is illustrated by Fig. 3. (a finally established SW is essentially discrete in this case). A worth noting transient feature which is apparent in this case is that the initial configuration gives rise to *two* SWs. One of them is originally characterized by the small size of the shock, but it very quickly evolves into a stable steep SW traveling at a constant velocity. The second SW reverses its propagation direction, and then quickly disappears. Lastly, an example of a strongly discrete case is shown in Fig. 4, when a very steep shock sets in almost instantaneously.

IV. VELOCITY OF THE SHOCK WAVE

The most essential characteristics of the established SW is the dependence of its velocity V and width W on the parameters ρ and α. Results demonstrating the $V(\rho)$ dependence are displayed in Fig. 5. The use of the estimates (10) shows that the lines pertaining to $\alpha=5$ and $\alpha=10$ belong to the essentially discrete region at $\rho>0.5$, and the line corresponding to $\alpha=1$ is inside the discrete region at $\rho>0.2$. An obvious inference suggested by Fig. 5 is that the dependence $V(\rho)$ is practically linear in the discrete region, resembling the linear dependence (9) in the continuum model. To explain this observation, we note that, as was concluded above, the strongly discrete model usually corresponds to large values of ρ. This implies that the linear conservative terms in Eq. (2) may be negligible in comparison with the much larger cubic conservative term; hence, dropping the conservative linear terms, the strongly discrete model may be simplified to a form in which its linear part is, effectively, *overdamped*,

$$i\dot{u}_n - i\alpha(u_{n+1}+u_{n-1}-2u_n) - |u_n|^2 u_n = 0. \qquad (11)$$

An evident property of Eq. (11), based on the scaling arguments, is that it gives rise to a SW velocity in the form

$$V_{\mathrm{discr}} = \alpha f(\rho/\sqrt{\alpha}), \qquad (12)$$

with an unknown function $f(\rho/\sqrt{\alpha})$. A straightforward comparison of Eq. (12) to the numerical results displayed in Fig. 5 suggests that the function f is very close to a linear one, $f(x)=C_{\mathrm{discr}}x$ with some constant coefficient C_{discr}, so that Eq. (12) becomes [cf. Eq. (9)]

$$V_{\mathrm{discr}} = C_{\mathrm{discr}}\sqrt{\alpha}\rho. \qquad (13)$$

TABLE I. Values of the empirical coefficient C_{discr} in Eq. (13), as found from the numerical data displayed in Fig. 5 for different values of α.

α	1	5	10	50
C_{discr}	1.6041	1.2027	1.1318	1.1244

The numerical data produce values of the coefficient C_{discr} displayed in Table I, which slightly depend upon α, for α taking values between 5 and 50, i.e., the simple semiempirical relation (13) holds well in this range. A considerable deviation in the case $\alpha=1$ in Table I is quite natural, as in this case neither α nor ρ (as per Fig. 5) are large enough to justify the use of the relation (12) pertaining to an effectively overdamped lattice. Equation (13) is also confirmed by Fig. 6, in which the velocity is shown as a function of the damp-

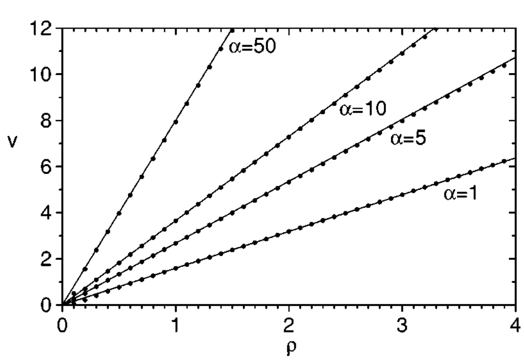

FIG. 5. Velocity of the shock wave found in the numerical simulations of Eq. (2) vs the asymptotic cw amplitude ρ at different fixed values of the dissipative constant α. Plotted quantities are dimensionless.

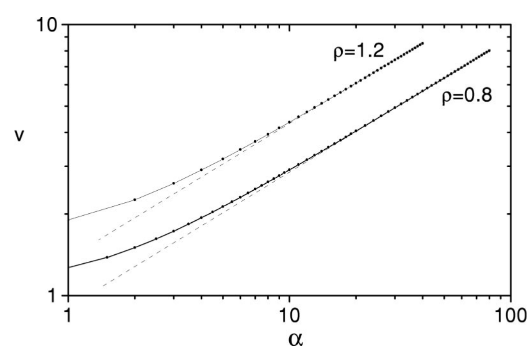

FIG. 6. Velocity of the shock wave vs the dissipative constant α on the double logarithmic scale at two fixed values of the cw amplitude, $\rho=0.8$ and $\rho=1.2$. According to the continuum-model estimate (10), the essentially discrete region in this figure is $\alpha<25$. Plotted quantities are dimensionless.

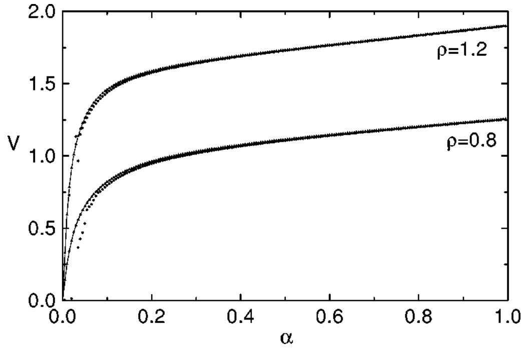

FIG. 7. The same as in Fig. 6, but at small values of α and on the usual scale. The triangles and circles show, respectively, the velocity defined as per Eq. (7) and as the distance traveled by the shock wave divided by the time. For this figure, the continuum-model estimate (10) predicts that the essentially discrete region is $\alpha > 0.4$. Plotted quantities are dimensionless.

ing for fixed values of ρ (note the logarithmic scales). From this figure we see that the velocity scales exactly as in Eq. (13) with $C_{\mathrm{discr}} \approx 1.12$, i.e., with the same coefficient as in Table I at large damping. Thus, a general conclusion is that the dependences $V(\rho)$ for the (effectively) overdamped model have a similar form in the discrete and continuum cases, cf. Eqs. (9) and (13), but with different numerical coefficients, $C_{\mathrm{contin}} \approx 0.75$ and $C_{\mathrm{discr}} \approx 1.12$.

The situation is quite different in the underdamped case. It is worth noting that, in this case, the velocity calculated as per Eq. (7) may be different from the straightforward one extracted from the simulations, i.e., the distance passed by SW divided by the time. The dependence of the SW's velocity, defined in both ways, upon the dissipative constant α at a fixed value of ρ is shown in Fig. 7 for the underdamped case. As is seen, the numerical and the power-balance velocity yield very close results for $\alpha > 0.3$, while at smaller α the power-balance expression in Eq. (7) yields results that appear to be much more natural than the ones obtained from numerical simulations (in the range of α shown in Figs. 5 and 6, the numerical and the power-balance velocity coincide). This discrepancy is due to the difficulty of measuring the numerical velocity at very small damping. In this region, indeed, the wave is very broad (see Fig. 8), and moves very slowly with a profile which fluctuates in time (see Fig. 1). An accurate measure of the velocity in this case would requires a very large system and a very long integration time, this leading to an unavoidable increase of the numerical error. Since in all other regions the agreement between these velocities is excellent, we believe that they would agree also at very small values of the dissipation parameter provided one could measure the velocity in the numerical experiment with adequate accuracy.

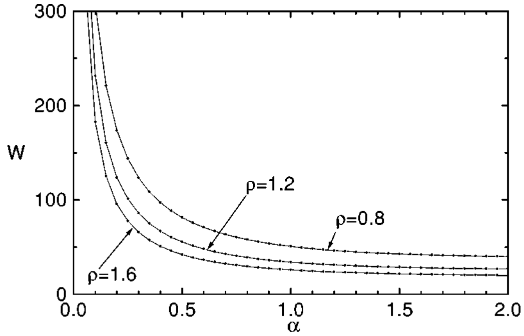

FIG. 8. Width of the shock wave vs the dissipative constant α at three fixed values of the cw amplitude, $\rho = 0.8$, $\rho = 1.2$, and $\rho = 1.6$. Plotted quantities are dimensionless.

An important inference following from the numerical results displayed in Fig. 7 is that both the numerical and the power-balance velocity vanish as α goes to zero. This is in apparent contradiction with Eq. (9), according to which in the continuum model the velocity keeps a nonzero value $V_{\mathrm{contin}} = \rho$ as $\alpha \to 0$. However, this constant value pertains, in the limit $\alpha \to 0$, to an infinitely long system which is clearly suggested by Eq. (10): the system's length must be much larger than $W_{\mathrm{contin}} \sim (\alpha\rho)^{-1}$, while the present numerical results were obtained for a system with a large but fixed size. In fact, numerical data presented in Ref. [9] show a trend for some decrease of the actual value of V_{contin} for small α, although this issue was not investigated in detail at very small values of α. On the other hand, it is very natural to expect that in *both* continuum and discrete large but finite systems, the SW velocity must indeed be vanishing in the limit $\alpha \to 0$, as in the absence of dissipation ($\alpha = 0$) there is no cause for the empty region to expand ousting the energy-carrying background. More accurately, we conjecture that, as the system's size is tending to infinity, the region of small α in which the velocity drops to zero is getting infinitely narrow. However, we did not check this conjecture in detail, as, being far from real physical applications, it requires very extensive simulations with high accuracy. Lastly, we notice that, as suggested by Fig. 7, the velocity can be regarded as being approximately constant for moderately small α, a steep fall to zero occurring only at very small values of α (see Fig. 7). Moreover, this roughly constant value is fairly close, for both values $\rho = 0.8$ and $\rho = 1.2$, to $\sqrt{2}\rho$, in accord with the prediction of the continuum model for (moderately) small α, see Eq. (9). Besides the velocity, it is also interesting to study a dependence of a SWs size on the model's parameters. In this connection, however, one should remember that a fairly smooth dependence of the width (unlike that for the velocity) is only possible in the quasicontinuum case. To this end, we have estimated the width W of the shock by computing the area underlying the modulus of its space derivative profile, and equating it to the area of a rectangle with basis W and height given by the maximum of the profile. We checked that in the quasicontinuum case this provides a good estimate of the width. At several fixed values of the background amplitude ρ (including those for which the velocity was displayed as a function of α in Figs. 5 and 7), we have collected data for the width in a region where it turns out to be a reasonably smooth function of the dissipative constant. These results are shown in Fig. 8 from which one can see that they are in good agreement with the prediction given by the upper part of the analytical estimate (10).

V. CONCLUSION

In this work we have studied shock waves, converting an energy-carrying domain into an empty one, in a discrete version of the normal-dispersion nonlinear Schrödinger equation with intrinsic losses, which can describe, e.g., an array of nonlinear optical fibers in a weakly lossy medium. Stable (receding) shock waves were found and compared to their counterparts in the earlier studied continuum model. In the overdamped case the dependence of the shock-wave's velocity on the amplitude of the energy-carrying background is characterized by a simple linear law, which differs only by a value of the numerical coefficient from a similar law in the continuum model. For the underdamped case, we have found that the velocity of the shock-wave is vanishing along with the loss constant α, which is in formal contradiction with the constant value of the velocity reported in Ref. [9] for the continuum model. The contradiction is explained by the fact that the latter value was actually predicted for an infinitely long system, for which the drop of the velocity to zero occurs at infinitely small α, so that the velocity remains nearly constant at small finite values of α. In fact, the numerical results obtained for the discrete model at moderately small α are in reasonable agreement with the analytical prediction of the continuum model. The dependence of the shock-wave's width on the dissipative constant was displayed, too, also being in good agreement with an analytical estimate.

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APPENDIX: NONEXISTENCE OF MOVING SHOCK WAVES IN THE DISSIPATIONLESS LIMIT

In this appendix we give an analytical argument which show that at zero damping ($\alpha=0$) SWs with nonzero velocity do not exist. To this end, we adopt the definition of the SW kinklike solution as $u_n(t) = U(n-Vt)\exp(-i\rho^2 t)$, where $|U|^2$ is a *monotonic* function of its argument, and V is the velocity. Then, at $\alpha=0$,

$$\sum_n [|u_n(t)|^2-|u_n(0)|^2]\equiv\sum_n [|U(n-Vt)|^2-|U(n)|^2]=0, \tag{A1}$$

as this sum is a conserved quantity according to Eq. (6), and it is zero at $t=0$. On the other hand, using an identity

$$V\int_0^{1/V}\sum_n f(n-Vt)\,dt=-\sum_n\int_n^{n-1}f(\xi)\,d\xi, \tag{A2}$$

we arrive at a relation

$$V\int_0^{1/V}\sum_n [|U(n-Vt)|^2-|U(n)|^2]dt =\sum_n \int_{n-1}^{n}[|U(\xi)|^2-|U(n)|^2]d\xi. \tag{A3}$$

Because the function $|U(\xi)|^2$ is assumed to be monotonic, the expression $[|U(\xi)|^2-|U(n)|^2]$ does not change its sign when ξ belongs to the interval $(n-1,n)$, hence the right-hand side of Eq. (A3) *cannot* be equal to 0, and, consequently, the expression on the left-hand side of the equation is not zero either. The latter inference is in direct contradiction with Eq. (A1), which proves that the discrete NLS-like models conserving the norm $\Sigma_n[|u_n(t)|^2-|u_n(0)|^2]$ cannot have solutions in the form of moving SWs. Note that other arguments in favor of the nonexistence of traveling kink solutions to discrete conservative NLS-like models were given in Ref. [14].

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